

Optimal Super–Oscillations

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Abstract

Super–oscillating signals are band limited signals that oscillate in some region faster than their largest Fourier component. While such signals have many scientific and technological applications, their actual use is hampered by the fact that an overwhelming proportion of the energy goes into that part of the signal, which is not super–oscillating. In the present article we consider the problem of optimization of such signals. The optimization that we describe here is that of the super–oscillation yield, the ratio of the energy in the super–oscillations to the total energy of the signal, given the range and frequency of the super–oscillations. The constrained optimization leads to a generalized eigenvalue problem, which is solved numerically. It is noteworthy that it is possible, to still increase the super–oscillation yield at the cost of slightly deforming the oscillatory part of the signal, while keeping the average frequency. We show, how this can be done gradually, which enables a trade-off between the distortion and the yield.

Index Terms

Super–oscillations, super–resolution, quantum theory, eigenvalues and eigenfunctions, time-frequency analysis.

Super oscillatory functions provide a stunning refutation of a very widely accepted lore, that band limited functions cannot oscillate with a frequency larger than its maximal Fourier component. A number of examples have been given in the past for such functions with very interesting applications to Quantum Mechanics [1]–[6], signal processing [7]–[11] and to optics, where super–oscillations are intimately related to super–resolution [12]–[17]. Interestingly, it was discovered that in random functions, defined as superpositions of plane waves with random complex amplitudes and directions, considerable regions

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are naturally super-oscillatory [18], [19]. Various mathematical aspects of the phenomenon have been discussed also more recently in [20], [21].

In a very important sense, though, the idea that a band limited function cannot oscillate faster than its largest Fourier component is not entirely false. It is well known that the super-oscillations exist in limited intervals of time (or regions of space, depending on the actual problem) and that the amplitude of the super-oscillations in those regions is extremely small compared to typical values of the amplitude in non-oscillating regions [2], [10], [11]. It is generically so small, that any hope of practical application of super-oscillating functions depends on tailoring the functions carefully to reduce that effect as much as possible. Two different approaches have been offered over the years to the problem of optimization of super-oscillation [7], [11], which will be discussed in relation to ours as we proceed.

In contrast to the two last references we prefer to work with periodic functions, which will thus be described by a finite number of Fourier coefficients. This choice, we believe has a number of advantages. First, it is clear, that it is more difficult to achieve super-oscillations, with a finite number of degrees of freedom than with an infinite number of degrees of freedom encoded in the Fourier **transform** of a band limited non-periodic function. Thus, achieving super-oscillations with a finite number of degrees of freedom is more challenging, yet enables a clear view of questions concerning the total number of oscillation versus the number of degrees of freedom. Second, we obtain an easy and practical way of constructing optimal super-oscillations.

Consider the function

$$f(t) = \frac{A_0}{\sqrt{2\pi}} + \sum_{m=1}^N \frac{A_m}{\sqrt{\pi}} \cos(mt) \quad (1)$$

Choose an interval $[0, a]$ with $a < \pi$ and impose on the function M constraints in the interval, $f(t_j) = \mu_j$ for $0 \leq t_j \leq a$ and $j = 0, \dots, M-1$. The constraints result in a set of M linear equations in $N+1$ unknowns of the form,

$$\sum_{m=0}^N C_{jm} A_m \equiv \mathbf{C}_j \cdot \mathbf{A} = \mu_j, \quad (2)$$

where

$$C_{jm} = \frac{1}{\sqrt{\pi}} \begin{cases} \cos(mt_j), & \text{for } m \neq 0 \\ 1/\sqrt{2}, & \text{for } m = 0 \end{cases}.$$

Generically this set of equations has no solution for $M > N+1$, has one solution for $M = N+1$ and a whole space of solutions for $M < N+1$. In particular, we can choose

$$t_j = \frac{aj}{M-1} \quad \text{and} \quad \mu_j = (-1)^j. \quad (3)$$

Provided $M \leq N + 1$, this choice constrains the function to oscillate within the interval $[-a, a]$ between the values ± 1 with a frequency

$$\omega = \frac{\pi(M-1)}{a}. \quad (4)$$

It is thus clear that the frequency of oscillation within the interval $[-a, a]$ can be increased indefinitely just by decreasing its size. Therefore, although to have a solution at all, we need that $M \leq N + 1$, the ratio between ω and N , the largest frequency appearing in the Fourier series, can be made as large as we want. Thus it is not a problem at all to obtain super-oscillations. This comes, at a cost, of course. First, we can obtain super-oscillations with a prescribed frequency ω only within an interval $[-a, a]$ with $a \leq \frac{\pi N}{\omega}$ and as stated before and will be demonstrated in the following (see Fig. 1 below) the amplitude in that region is relatively extremely small.

Next, we would like to optimize our super-oscillating function for fixed a and $M < N + 1$ but we have to decide first in what sense do we want to optimize it. Ferreira and Kempf [11], consider the energy of the signal, $E = \int_{-\infty}^{\infty} f^2(t)dt$, use the fact that f is band limited and minimize the energy under the interpolation constraints (equation (3)). We believe that for many applications, the right quantity to maximize under the constraints is the super-oscillation yield,

$$Y(M, a) = \frac{\int_{-a}^a f^2(t)dt}{\int_{-\infty}^{\infty} f^2(t)dt}, \quad (5)$$

rather than the total energy. (Note that as will become evident in the following the yield is not just a function of ω but of M and a separately. For the discrete case described in (1), we take instead of the energy, which is infinite, the energy per period. Thus the super-oscillation yield that we maximize under the constraints is

$$Y(N, M, a) = \frac{\int_{-\pi}^{\pi} f^2(t)dt}{\int_{-\pi}^{\pi} f^2(t)dt} = \frac{\sum_{m,n=0}^N \Delta_{mn} A_m A_n}{\sum_{m=0}^N A_m^2} \equiv \frac{I}{D}, \quad (6)$$

where the entries of the matrix Δ are given by

$$\Delta_{mn} = \begin{cases} \frac{2m \cos(na) \sin(ma) - n \cos(ma) \sin(na)}{\pi(m^2 - n^2)}, & m \neq n \neq 0 \\ \frac{1}{\pi} \left(a + \frac{\sin(2na)}{2n} \right), & m = n \neq 0 \\ \frac{\sqrt{2}}{\pi n} \sin(na), & m = 0, n \neq 0 \\ \frac{a}{\pi}, & m = n = 0 \end{cases}. \quad (7)$$

The set of M vectors $\{\mathbf{C}_j\}$ defined in (2) spans an M dimensional space. In this space we introduce an orthonormal basis $\{\hat{e}_{N-M+1}, \dots, \hat{e}_N\}$. An orthonormal basis for the full N dimensional vector space is then constructed by adding the set $\{\hat{e}_0, \dots, \hat{e}_{N-M}\}$, such that $\hat{e}_i \cdot \hat{e}_j = \delta_{ij}$ for all $i, j = 0, \dots, N$ (up to the orthogonality requirement, the basis vectors can be chosen randomly, of course). We obtain thus the rotated degrees of freedom,

$$B_i = \hat{e}_i \cdot \mathbf{A}, \quad (8)$$

with the obvious advantage that the last M degrees of freedom in \mathbf{B} are constrained independently of each other and are equal to linear combinations of the μ_j 's. Thus we denote

$$B_i = \tilde{\mu}_i \quad \text{for } i = N - M + 1, \dots, N. \quad (9)$$

The numerator I in (6) can be written now in terms of the rotated degrees of freedom B as $I = \sum_{m,n=0}^N \Delta_{mn} A_m A_n = \sum_{m,n=0}^N \Delta_{mn}^{(R)} B_m B_n$, where $\Delta^{(R)} = \mathbf{R} \Delta \mathbf{R}^{-1}$, \mathbf{R} being the rotation that takes \mathbf{A} into \mathbf{B} . Let us describe the matrix $\Delta^{(R)}$ by the following block form

$$\Delta^{(R)} = \begin{pmatrix} \tilde{\Delta}_{(N+1-M) \times (N+1-M)} & \Gamma_{(N+1-M) \times M} \\ \hat{\Gamma}_{M \times (N+1-M)} & \bar{\Delta}_{M \times M} \end{pmatrix}, \quad (10)$$

where $\hat{\Gamma}$ is the transpose of Γ . The super-oscillation yield expressed in terms of the unconstrained B 's is

$$Y = \frac{\sum_{m,n=0}^{N-M} \tilde{\Delta}_{mn} B_m B_n + 2 \sum_{m=0}^{N-M} \sum_{n=N+1-M}^N \Gamma_{mn} \tilde{\mu}_n B_m + \sum_{m,n=N+1-M}^N \bar{\Delta}_{mn} \tilde{\mu}_m \tilde{\mu}_n}{\sum_{m=0}^{N-M} B_m^2 + \sum_{m=N+1-M}^N \tilde{\mu}_m^2}. \quad (11)$$

Differentiating the yield with respect to B_m and equating to zero yields

$$\mathbf{B} = -(\tilde{\Delta} - \mathbf{Y}\mathbf{I})^{-1} \Gamma \tilde{\mu}, \quad (12)$$

where \mathbf{I} is the unit matrix and $\tilde{\mu}$ is the vector of the $\tilde{\mu}_j$'s. It is thus clear, that the components of the vector \mathbf{B} depend on I and on D only through the ratio $Y = \frac{I}{D}$. Those components are, by Cramers rule, a ratio of two determinants. The determinant in the denominator is clearly a polynomial of degree $N + 1 - M$ in Y . For each entry of \mathbf{B} the determinant in the numerator is that of the matrix, obtained from $(\tilde{\Delta} - Y\mathbf{I})$ by replacing one of the columns by the vector $(\Gamma \tilde{\mu})$. Therefore, B_m , the m 's component of the vector \mathbf{B} , for which the yield is extremal, is given by the explicit expression

$$B_m = \frac{P_m^{N-M}(Y)}{P^{N+1-M}(Y)}, \quad (13)$$

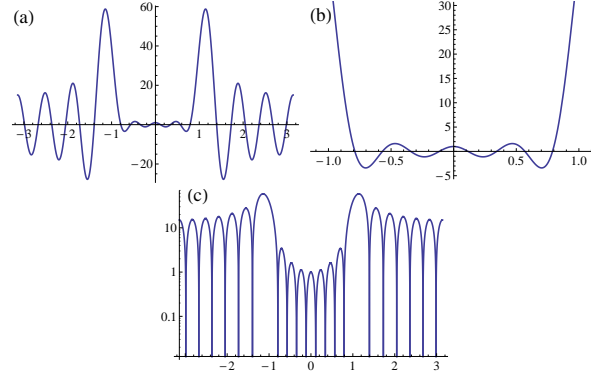


Fig. 1. The super-oscillating signal for $N = 10$, $M = 5$ and $a = 1$ **(a)** on a linear scale in $(-\pi, \pi)$, **(b)** on a linear scale focusing on the segment $(-1, 1)$, and **(c)** on a semi-logarithmic scale.

where $P_m^{N-M}(Y)$ is a polynomial of degree $N - M$ in Y and $P^{N+1-M}(Y)$ is a polynomial of degree $N + 1 - M$. Clearly, an equation for the super-oscillation yield can be obtained by plugging the right hand side of (12) directly into the right hand side of (10). We prefer, though, a different route that yields a more simplified form for the equation. The expression $C = D^2 \sum_{m=0}^N B_m \frac{\partial(I/D)}{\partial B_m}$ is identically zero by the extremum condition. This results in a simplified equation for Y ,

$$\sum_{m=0}^{N-M} \sum_{n=1}^M \Gamma_{mn} \tilde{\mu}_n P_m^{N-M}(Y) = \left(\sum_{n=1}^M \tilde{\mu}_n^2 \right) Y P^{N+1-M}(Y) - \left(\sum_{m,n=1}^M \bar{\Delta}_{mn} \tilde{\mu}_m \tilde{\mu}_n \right) P^{N+1-M}(Y), \quad (14)$$

which is a polynomial equation of degree $N + 2 - M$. We view each of the solution as a generalized eigenvalue, as to each of the roots corresponds a $N + 1 - M$ dimensional vector. (In contrast to the traditional eigenvalue problem, the eigenvectors are determined by the inhomogeneous linear equation (11) and the number of generalized eigenvalues is $N + 2 - M$, and those determine $N + 2 - M$ generalized eigenvectors. This means, of course, that the set of generalized eigenvectors is linearly dependent.) We are naturally interested in the largest generalized eigenvalue that corresponds to the maximal super-oscillating yield. The generalized eigenvector corresponding to that solution has in the interval $[-a, a]$ the exact super-oscillation frequency imposed by the constraints. In Fig. 1 we present the super-oscillating signal for $N = 10$, $M = 5$ and $a = 1$.

A natural question to ask is: can we learn anything from generalized eigenvectors corresponding to lower generalized eigenvalues? In Fig. 2 we present the super-oscillating portion of the signal corresponding to various generalized eigenvalues for $N = 10$, $M = 6$ and $a = 1$. It is obvious that as we go to

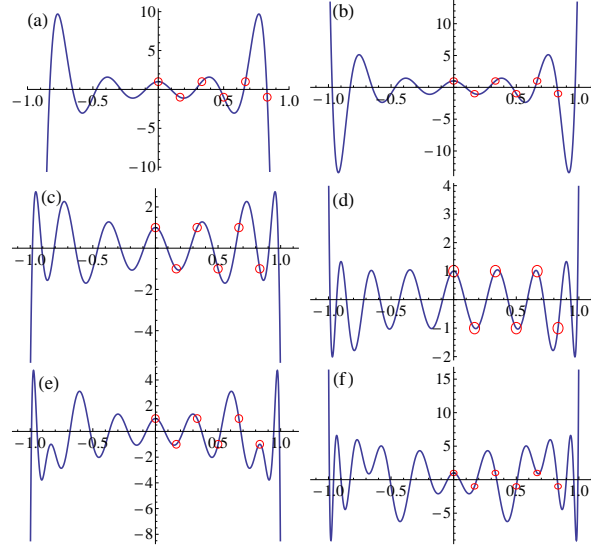


Fig. 2. The super-oscillating portion of signals corresponding to the various generalized eigenvalues, with λ_6 (figure (a)) being the maximal eigenvalue, which is the maximal yield and λ_1 (figure (f)) the smallest. The red circles represent the constrained points. The eigenvalues are (a) $\lambda_6 = 0.00233$, (b) $\lambda_5 = 4.847 \times 10^{-7}$, (c) $\lambda_4 = 1.379 \times 10^{-10}$, (d) $\lambda_3 = 2.559 \times 10^{-14}$, (e) $\lambda_2 = 2.176 \times 10^{-18}$, (f) $\lambda_1 = 1.189 \times 10^{-23}$. As can be seen, every time we switch to a smaller eigenvalue another oscillation inside the super-oscillating region appears.

lower generalized eigenvalues the number of oscillations in the super-oscillating interval keeps increasing, adding more oscillations to those imposed by the constraints. In fact the number of oscillations in the interval $[-a, a]$ grows exactly by one when we go from a generalized eigenvalue λ_i to the one immediately below it $\lambda_{i-1} < \lambda_i$. The number of oscillations inside $[-a, a]$ corresponding to the generalized eigenvalue λ_i is thus exactly $N + 1 - i$ ($1 \leq i \leq N + 2 - M$).

How is $\lambda_i(N, M, a)$ related to the maximal generalized eigenvalue, $Y(N, M, a)$, for the case with the same N and a but with a **constrained** number of oscillations equal to the actual number of oscillation corresponding to λ_i , $N + 1 - i$? The answer seems intuitively clear. Obtaining the same number of oscillations, while imposing less constraints, is expected to give a higher yield. Namely, we expect that for every $M' > M$ and $1 \leq i \leq N + 2 - M'$ it will hold that $\lambda_i(N, M, a) \geq \lambda_i(N, M', a)$. In particular since $Y(N, M, a) = \lambda_{N+2-M}(N, M, a)$ we also have $\lambda_i(N, M, a) \geq Y(N, N + 2 - i, a)$. This intuitive feeling, is obviously exact, in the case where the set of M constraints, $S(M)$ obeys $S(M) \subset S(N + 2 - i)$. To complete the picture for cases where $S(M) \not\subset S(N + 2 - i)$, we give more details in Figs. 3 and 4. In Fig. 3 we give the different eigenvalues for fixed N and M as a function of a . In Fig. 4 we give the eigenvalue for fixed a and N as a function of i for various values of M . This will give not just an

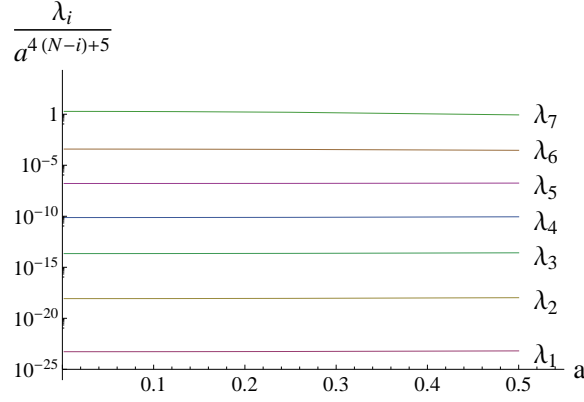


Fig. 3. The dependence of the eigenvalue λ_i on a for $a < 0.5$, fixed $N = 10$ and $M = 5$.

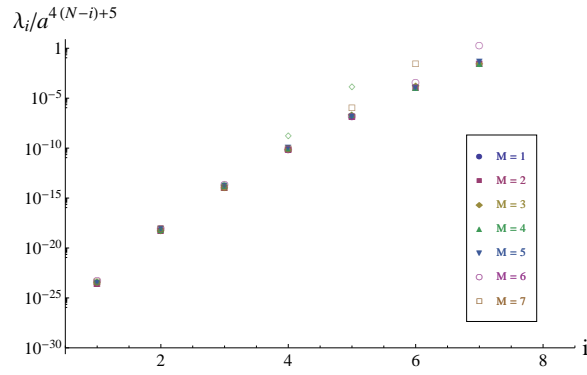


Fig. 4. The dependence of the eigenvalue λ_i on the index i for fixed $N = 10$, $a = 1/64$ and various M 's.

inequality but a full quantitative picture for small (not necessarily very small) a .

The improvement in the yield for a given number of oscillations in the super-oscillation region, obtained by decreasing i comes at a price, though. The seemingly periodic structure near the middle of the super-oscillation interval, corresponding to the highest eigenvalue, generically deteriorates as the number of super-oscillations increases. This suggests a possible trade-off between the quality of super-oscillation and the yield. For that trade-off, we have at our disposal a whole spectrum of signals, with the same number of super-oscillations, corresponding to the same number of low Fourier components. On one side of the spectrum we find the signal where all the oscillations are constrained, such that the quality of the super-oscillation is good but the yield is relatively low. On the other side of the spectrum we find the function with $M = 1$. For that function we only fix the value of f at the origin to be 1. This imposes no oscillation at all on the function in the super-oscillating region. Since, optimal f 's, which are

necessarily symmetric, are not expected to vanish at the origin, this implies that we are not constraining the function at all and the maximization of the yield is equivalent to maximizing it under the requirement that the total energy is normalized. This will yield a set of $N + 1$ ordinary eigenvalues and eigenvectors. In the corresponding continuum problem the number of eigenvalues is infinite and the eigenvectors are the prolate spheroidal wave functions of Slepian and Pollak [7]. (Ref. [9] mentions a discrete case but it is a totally different discreteness than that we study, i.e., in equation (1).) It is interesting to note that the signals obtained by the discrete Ferreira-Kempf procedure [11] do not belong to the family of functions described above. Those signals are obtained by minimizing our denominator (defined in (6)),

$$D = \sum_{m=0}^N A_m^2 = \sum_{m=0}^{N-M} B_m^2 + \sum_{m=N+1-M}^N \tilde{\mu}_m^2, \quad (15)$$

and under the same oscillation constraints we use (i.e., equation (3)). It is clear thus that those eigenvectors are obtained by simply setting

$$B_m = 0 \quad \text{for} \quad m = 0, \dots, N - M. \quad (16)$$

To conclude, we have shown in this paper how to obtain optimal super-oscillating signals that allow a gradual trade-off between super-oscillation yield and quality of the signal. Since our optimization process is based on a specific way of constraining the signal to produce super-oscillations, given by equation (3), improvements of the yield and/or the super-oscillation quality may be expected and will be discussed in future publications.

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